Probabilistic Method and Random Graphs

Lecture 11. De-randomization and Sample&Modify

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[1]The slides are mainly based on Chapter 6 of Probability and Computing.

Comments, questions, or suggestions?

A Review of Lecture 10

- Principle of probabilistic method

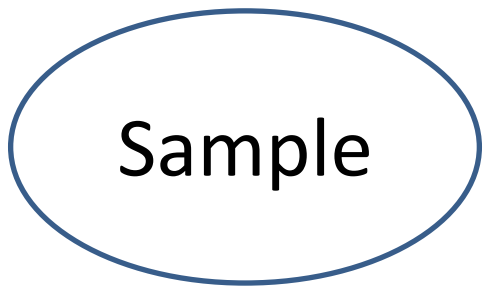
Probability space of all candidates → Nonzero probability of satisfying the property → Existence

- Counting: Tournament, Ramsey number
- First moment method: Max-3SAT, MIS
  - Expectation argument: $\Pr(X \geq \mathbb{E}[X]) > 0, \Pr(X \leq \mathbb{E}[X]) > 0$
  - Markov's inequality: $\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$

$$\Pr(X \neq 0) = \Pr(X > 0) = \Pr(X \geq 1) \leq \mathbb{E}[X]$$

A Review of Lecture 10

- How to find a desirable object? By sampling!
- Algorithmic paradigm



A Review of Lecture 10

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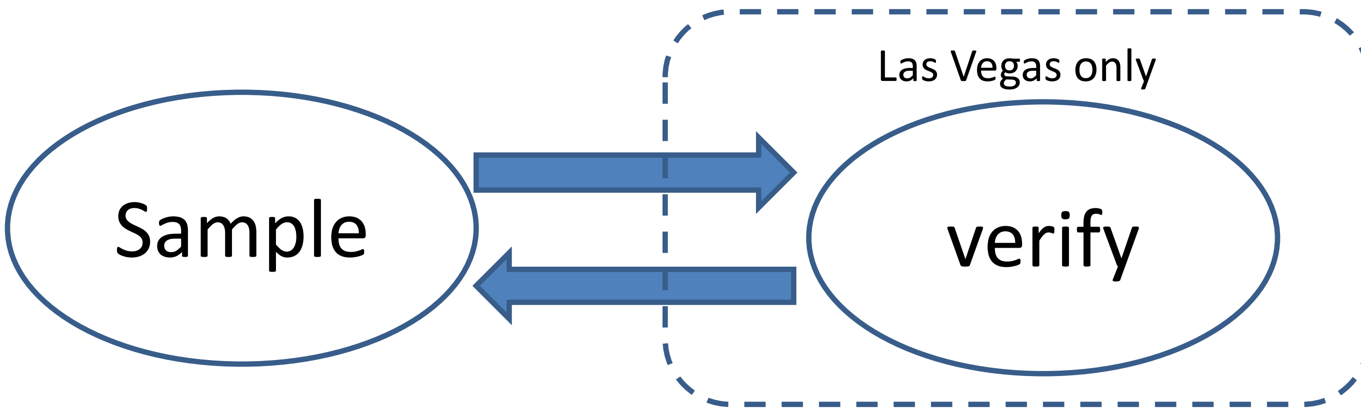


- First moment method guarantees efficiency

Expectation argument

- Turán Theorem
  - Any graph G=(V,E) contains an independent set of size at least $\frac{|V|}{D+1}$, where $D = \frac{2|E|}{|V|}$
- Proof: Consider the following random process for constructing an independent set S:
  - Initialize S to be the empty set
  - Repeat: Remove S and its neighbors; randomly choose a remaining vertex u
  - Return S

Proof (Continued)

- S is an independent set
- Vertex u is selected with probability $\geq \frac{1}{d(u)+1}$
  - See the next slide
- So, $\mathbb{E}[|S|] \geq \sum \frac{1}{d(u)+1} \geq \frac{|V|}{D+1}$ due to convexity

- **Remark:** probability of sampling a good independent set is $\geq \frac{1}{2D|V|^2}$

Proof: Pr(u is selected)$\geq \frac{1}{d(u)+1}$

- u is selected if and only if A occurs
  - A: when sampling first occurs in the neighborhood of u, u rather than its neighbors is sampled
    - Neighborhood: u and its then-valid neighbors
  - Denote the neighborhood by N, and the number of then-valid neighbors by x. Note that $x \leq d(u)$
- $\Pr(A) = \Pr(u$ is chosen$|$sampling occurs in $N)$

$$= \frac{1}{x+1} \geq \frac{1}{d(u)+1}$$

- Cool to get an efficient randomized algorithm

- Can we derive a deterministic one?

- Yes, if expectation argument is used

De-randomization: an example

- **MAX-3SAT**: Given a 3-CNF Boolean formula, find a truth assignment satisfying the maximum number of clauses
  - E.g.: $(x_1 \vee x_2 \vee x_3) \wedge \ldots \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$
- Known: at least $\frac{7}{8}n$ clauses can be satisfied
- Randomized algo. to find a good assignment
  - Independently, randomly assign values to variables
  - Succeed if lucky
    - Can we make good choice, rather than pray for luck?

Look closer at the randomized algorithm

- In equivalence, choose values sequentially
- Good choices lead to a good final result
  - Which choice is good?
    - Easy to know with hindsight, but how to predict
  - A tentative approach: always make the choice which allows a good final result
    - Fact: a $\frac{7n}{8}$ expect. means the existence of a $\frac{7}{8}$-approx.
    - Make the current choice, keeping the expectation $\geq \frac{7n}{8}$
  - Nice, but does such a choice exist? How to find it?

Conditional expectation says yes!

- The first step
  - $\frac{7n}{8} = \mathbb{E}[X] = \boxed{\sum_{v_1} \Pr(x_1 = v_1)} \mathbb{E}[X | x_1 = v_1]$
  - There must be v_1 s.t. $\mathbb{E}[X | x_1 = v_1] \geq \frac{7n}{8}$
- Likewise, if $\mathbb{E}[X | x_1 = v_1, \dots, x_{k-1} = v_{k-1}] \geq \frac{7n}{8}$, then $\mathbb{E}[X | x_1 = v_1, \dots, x_k = v_k] \geq \frac{7n}{8}$ for some v_k
- Final correctness
  - $X(x_1 = v_1, \dots, x_m = v_m) = \mathbb{E}[X | x_1 = v_1, \dots, x_m = v_m] \geq \frac{7n}{8}$
- Given $v_1, \dots, v_{k-1}$, what's the v_k?
  - Let v_k s.t. $\mathbb{E}[X | x_1 = v_1, \dots, x_k = v_k]$ is maximized

Deterministic $\frac{7}{8}$-algorithm for MAX-3SAT

For $k = 1 \cdots m$ do

Assign to x_k the value v_k that maximizes

$$\mathbb{E}[X | x_1 = v_1, \ldots x_{k-1} = v_{k-1}, x_k = v_k]$$

Endfor

- Cool! And this approach can be generalized

De-randomization via conditional expectation

- Expectation argument$\Longrightarrow$deterministic algorithm
- Basic idea
  - Expectation argument guarantees existence
  - Sequentially make deterministic choices
    - Each choice maintains the expectation, given the past ones

- Only valid for expectation argument where randomness lies in a sequence of random variables
- What if the expectation is hard to compute?

Example: Turán Theorem

- Any graph $G = (V, E)$ contains an independent set of size at least $\frac{|V|}{D+1}$, where $D = \frac{2|E|}{|V|}$
- Expectation argument: the expected size of an independent set S is at least $\frac{|V|}{D+1}$
- Randomly choose vertices into S one by one

- Try the de-randomization routine

Idea of the algorithm (1)

- Choose valid vertices sequentially
- At step $t+1$, find u to maximize $\mathbb{E}\left[Q|S^{(t)},u\right]$
  - $S^{(t)}$: the independent set at step t
  - Q: the size of the final independent set
- Hard to compute the expectation ☹
  - $\mathbb{E}[Q] \geq \sum \frac{1}{d(w)+1} \geq \frac{|V|}{D+1}$
- It suffices to show $\mathbb{E}\left[Q|S^{(t)}\right] \geq \frac{|V|}{D+1}$ for any t

Idea of the algorithm (2)

- Note that $\mathbb{E}\left[Q|S^{(t)}\right] \geq \left|S^{(t)}\right| + \sum_{w \in R^{(t)}} \frac{1}{d(w)+1} \triangleq X^{(t)}$
  - $R^{(t)}$: set of vertices out of the neighborhood of $S^{(t)}$
- $X^{(0)} \geq \frac{|V|}{D+1} \Rightarrow$ it's enough if $X^{(t)}$ is non-decreasing
  - Can we achieve this?
- If at step $t+1$, $u \in R^{(t)}$ is chosen,
  $X^{(t+1)} - X^{(t)} = 1 - \sum_{w \in \Gamma^{+}(u)} \frac{1}{d(w)+1}$

Can it be non-negative?

- $\sum_{u \in R^{(t)}} \left(1 - \sum_{w \in \Gamma^{+}(u)} \frac{1}{d(w)+1}\right) \geq \left|R^{(t)}\right| - \sum_{w \in R^{(t)}} \frac{d(w)+1}{d(w)+1} = 0$
- So, there is u s.t. $X^{(t+1)} \geq X^{(t)}$
  - Any $u \in R^{(t)}$ that minimizes $\sum_{w \in \Gamma^{+}(u)} \frac{1}{d(w)+1}$ works

A deterministic algorithm

- Initialize S to be the empty set
- **While** there is a vertex $u \notin \Gamma(S)$
  - Add to S such a vertex u which minimizes $\sum_{w \in \Gamma^+(u)} \frac{1}{d(w)+1}$
- **Return** S

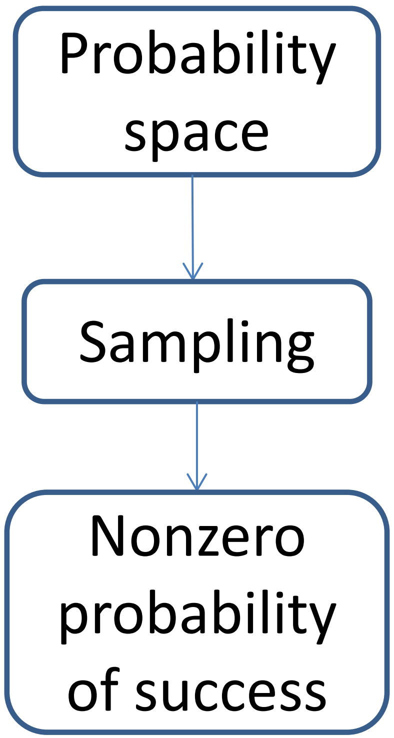
- Paul Turán (1910 –1976)
- Hungarian mathematician
- Founder of

  Probabilistic number theory

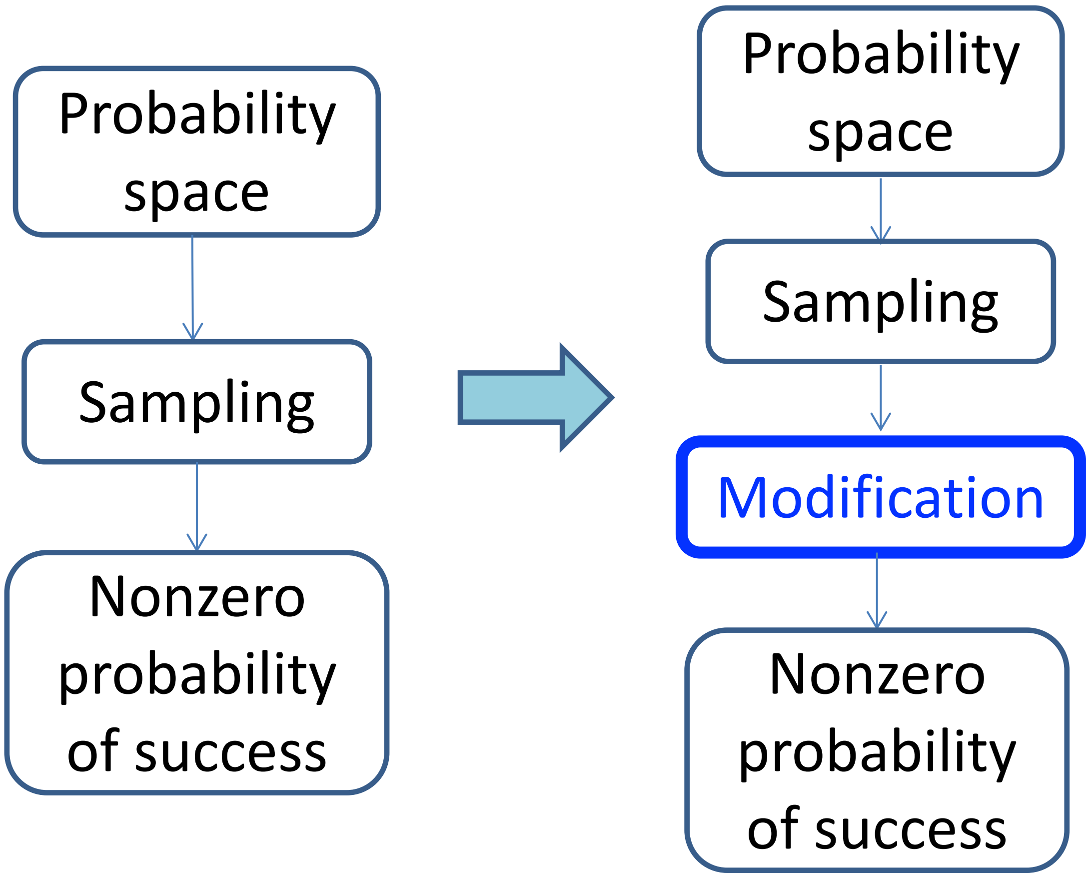
  Extremal graph theory

  (in Nazi Camp)

Sample


Probability space
Sampling
Nonzero probability of success

Sample and Modify



Big Chromatic Number and Big Girth

- Chromatic number vs local structure
  - Sparse local structure → small chro. number?
  - No! (Erdős 1959)
- One of the first applications of prob. Method
- Theorem: for any integers $g, k > 0$, there is a graph with girth$\geq g$ and chro. number$\geq k$
- We just prove the special case $g = 4$, i.e. triangle-free

Basic Idea of the Proof

- Randomly pick a graph G from $G_{n,p}$
  - $\chi(G)$: the chromatic number of G
  - $\mathbb{I}(G)$: the size of a maximum independent set of G
- With high probability $\mathbb{I}(G)$ is small
  - $\mathbb{I}(G)\chi(G) \geq n$ implies that $\chi(G)$ is big
- With high probability G has few triangles
- Destroy the triangles while keeping $\mathbb{I}(G)$ small

Proof: $\mathbb{I}(G)$ is small w.h.p.

- S: a vertex set of size $\frac{n}{2k}$
- A_S: S is an independent set
- $\Pr\left(\mathbb{I}(G) \geq \frac{n}{2k}\right) = \Pr(\bigcup_S A_S)$

$$\leq \binom{n}{n/2k}(1-p)^{\binom{n/2k}{2}}$$

$$< 2^n e^{-\frac{pn(n-2k)}{8k^2}}$$

which is small if n is large and $p = \omega(n^{-1})$

Proof: triangles are few w.h.p.

- $\mathcal{T}(G)$: the number of triangles of G
- $\mathbb{E}[\mathcal{T}(G)] = \binom{n}{3}p^3 < \frac{(np)^3}{6} = \frac{n}{6}$ if $p = n^{-\frac{2}{3}}$
- By Markov ineq., $\Pr\left(\mathcal{T}(G) > \frac{n}{2}\right) \leq \frac{1}{3}$
- Recall $\Pr\left(\mathbb{I}(G) \geq \frac{n}{2k}\right) < 2^n e^{-\frac{pn(n-2k)}{8k^2}}$

$$< e^n\, e^{-\frac{pn^2}{16k^2}} = e^{n - n^{\frac{4}{3}}/16k^2} \quad \text{if } n > 4k$$

$$< e^{-n} < \frac{1}{6} \quad \text{if } n^{1/3} \geq 32k^2$$

Proof: modification

- $\Pr\left(\mathbb{I}(G) < \frac{n}{2k}, \mathcal{T}(G) \le \frac{n}{2}\right) > \frac{1}{2}$
  - Choose G s.t. $\mathbb{I}(G) < \frac{n}{2k}, \mathcal{T}(G) \le \frac{n}{2}$
- Remove one vertex from each triangle of G, resulting in a graph G' with $n' \ge n - \mathcal{T}(G)$
- $\mathbb{I}(G') \le \mathbb{I}(G) < \frac{n}{2k}$
- $\chi(G') \ge \frac{n'}{\mathbb{I}(G')} \ge \frac{n-\mathcal{T}(G)}{\frac{n}{2k}} \ge k$

Algorithm for finding such a graph

- Fix $n^{1/3} \geq 32k^2$ and $p = n^{-2/3}$
- Sample G from $G_{n,p}$
- Destroy the triangles

- Success probability > ½

- Do you have any idea of de-randomizing?

References

- http://www.cse.buffalo.edu/~hungngo/classes/2011/Spring-694/lectures/sm.pdf
- http://www.openproblemgarden.org/
- Documentary film of Erdős: N is a Number - A Portrait of Paul Erdős

Thank you!