Homework of Week 5

Deadline: 9:00am, October 28 (Thursday), 2021

1. Prove Chernoff-like bounds for Poisson random variable X_μ with expectation μ:

   (a) If $x > \mu$, then $Pr(X_\mu \geq x) \leq \frac{e^{-\mu}(e\mu)^x}{x^x}$

   (b) If $x < \mu$, then $Pr(X_\mu \leq x) \leq \frac{e^{-\mu}(e\mu)^x}{x^x}$

2. (**Optional. Bonus score 5 points**) Prove the Poisson convergence theorem with weak dependence. Namely, for each n, suppose random variables $X_1^n, ..., X_n^n \in \{0, 1\}$ satisfy

   - $\lim_{n\to\infty} \mathbb{E}[Y_n] = \lambda$ where $Y_n = \sum_{i=1}^n X_i^n$, and
   - For any k, $\lim_{n\to\infty} \sum_{1\leq i_1<...<i_k\leq n} \Pr\left(X_{i_1}^n = X_{i_2}^n = ... = X_{i_r}^n = 1\right) = \lambda^k/k!$

   Then $\lim_{n\to\infty} Y_n \sim Poi(\lambda)$, i.e. $\lim_{n\to\infty} \Pr(Y_n = k) = e^{-\lambda}\lambda^k/k!$ for any integer $k \geq 0$. (Hint: you may need Bonferroni inequalities)

3. Let X be a Poisson random variable with mean μ, representing the number of errors on a page of this book. Each error is independently a grammatical error with probability p and a spelling error with probability $1 - p$. If Y and Z are random variables representing the numbers of grammatical and spelling errors (respectively) on a page of this book, Prove that Y and Z are Poisson random variables with means $p\mu$ and $(1 - p)\mu$, respectively. Also, prove that Y and Z are independent.

4. The following problem models a distributed system wherein agents contend for resources but *back off* in the face of contention. Balls represent agents, and bins represent resources.

   The system evolves over rounds. Every round, balls are thrown independently and uniformly at random into n bins. Any ball that lands in a bin by itself is served and removed from consideration. The remaining balls are thrown again in the next round. We begin with n balls in the first round, and we will finish when every ball is served.

   - If there are b balls at the start of a round, what is the expected number of balls at the start of the next round?
   - Suppose that in every round the number of balls served is exactly the expected number of balls to be served. Show that all the balls would be served in $O(\ln\ln n)$ rounds. (Hint: If x_j is the expected number of balls left after j rounds, show and use that $x_{j+1} \leq x_j^2/n$.)

5. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1s_2...s_i...s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.