

## Homework of Week 3

### Deadline: 9:00am, October 14 (Thursday), 2021

1. Prove the following extensions of the Chernoff bound. Let  $X = \sum_{i=1}^n X_i$ , where the  $X_i$ 's are independent Poisson trials. Let  $\mu = \mathbb{E}[X]$ . Choose any  $\mu_L$  and  $\mu_H$  such that  $\mu_L \leq \mu \leq \mu_H$ . Then, for any  $\delta > 0$ ,  $\Pr(X \geq (1 + \delta)\mu_H) \leq \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^{\mu_H}$ .

Similarly, for any  $0 < \delta < 1$ ,  $\Pr(X \leq (1 - \delta)\mu_L) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu_L}$ .

2. Let  $X_1, \dots, X_n$  be independent Poisson trials such that  $\Pr(X_i = 1) = p_i$  and let  $a_1, \dots, a_n$  be real numbers in  $[0, 1]$ . Let  $X = \sum_{i=1}^n a_i X_i$  and  $\mu = \mathbb{E}[X]$ . Then the following Chernoff bound holds: for any  $\delta > 0$ ,  $\Pr(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu$ . Also prove a similar bound for the probability  $\Pr(X \leq (1 - \delta)\mu)$  for any  $0 < \delta < 1$ .

3. A function  $f$  is said to be convex if it holds that  $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$  for any  $x_1, x_2$  and  $0 \leq \lambda \leq 1$ .

- Let  $Z$  be a random variable that takes on a finite set of values in  $[0, 1]$ , and let  $p = \mathbb{E}[Z]$ . Define the Bernoulli random variable  $X$  by  $\Pr(X = 1) = p$  and  $\Pr(X = 0) = 1 - p$ . Show that  $\mathbb{E}[f(Z)] \leq \mathbb{E}[f(X)]$  for any convex function  $f$ . (Hint: induce on the number of values that  $Z$  takes on, and apply the convexity.)
- Use the fact that  $f(x) = e^{tx}$  is convex for any fixed  $t \geq 0$  to obtain a Chernoff-like bound for  $Z$ .

4. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string  $s_1 s_2 \dots s_{20}$ , where  $s_i$  is 1 if the  $i^{\text{th}}$  trial gets Head, and otherwise is 0.